The Telegrapher's Equation

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Introduction

This article deals with an electrical transmission line with constant linear parameters R, L, C and G, in both the frequency and time domains. Among the examples will be a typical telegraph line in the days of iron wires, the coaxial cable of the first successful Atlantic telegraph, and twisted-pair telephone cables, with parameters estimated as best I can. These examples are very similar to most actual transmission lines, and will illustrate the principles quite well. We will not go into the interesting mathematical questions very deeply. For this, the reader is referred to the References.

The overhead aerial line was used by Steinheil in Munich around 1838, and was applied by Wheatstone and Cooke in their telegraph from Paddington to Slough around 1840, after experiments with buried lines proved them unsuitable (at the time; they are now general). The Morse patentees tried buried lines as well, which failed utterly, but Vail and Cornell quickly introduced aerial lines in 1844 in the first U.S. commercial line from Washington, D.C. to Baltimore. The original insulated copper wire was soon replaced by larger iron wire for strength, with little increase in resistance. The wire was left bare as soon as Vail was sure the electricity would not leak out. The copper wire at that time, used mainly by haberdashers, would have had little better than 40% conductivity, making the resistivity of iron only about twice as great. A typical line of bare galvanized iron wire was supported on glass insulators in the U.S. or on porcelain insulators in Europe with about 40 poles per mile. A ground return was used. If well-built, such a line is of high electrical quality.

The English Channel between Dover and Calais was crossed by a submarine cable in 1851, linking Britain to the continental telegraph system. This 25-mile cable was laid in a maximum depth of 30 fathoms. It consisted of a core of copper wires, an insulating sheath of gutta-percha, and an armor of iron wires for strength and protection, similar to later cables. Gutta-percha was a natural plastic chemically related to rubber, with a very high resistivity and a dielectric constant of 2.6. It has now been replaced by polyethylene (PE) and other similar polymers. Gutta-percha survives in an ocean-water environment. In fact, cables using it must be kept wet from the time of manufacture.

Gutta-percha is probably best known as the material from which golf balls are (or were) made. A whitish, tough material, it comes from the sap of the percha tree of Malaya, of the family Sapotaceae. It was brought to England by Dr Montgomerie of the East India Company in the 1840's. In 1848, Hancock invented a machine for coating wire with gutta-percha, since its value as an insulator was instantly recognized, and there was a great need for it, and plastics were then unknown. In contact with the air, gutta-percha oxidized and perished. This defect was cured by vulcanization around 1850, but then the sulphur in the vulcanized product attacked copper wires. However, when kept submerged, gutta-percha did not deteriorate, and was a valuable insulator for submarine cables. The Gutta-Percha Co. was organized for its exploitation. The lack of a good insulator had rendered buried cables impractical to this time, as Morse and Wheatstone had found to their dismay. Gutta-percha is *trans*-polyisoprene (C_5H_8), while rubber is *cis*-polyisoprene, only a structural difference. It is not elastic, like rubber, but is flexible enough for use in cables.

Ireland was connected in 1853 by a cable from Portpatrick in Scotland to Donaghadie, in 180 fathoms, which was more difficult and required several tries. Thereafter, much was learned about cable laying in projects of greater and greater ambition as the world's telegraph network was assembled. The Anglo-American Atlantic Telegraph Company was formed in 1856 with money from Cyrus Field and technical expertise from Britain. The cable was projected to extend from Valentia Bay in Ireland to Trinity Bay in Newfoundland, 3039 km, in depths of 1700-2400 fathoms, which was by far the most ambitious project yet attempted. The project was rushed to completion in 1857, but the first attempt in that year quickly failed. The two companies independently manufacturing halves of the cable managed to spiral the armor in opposite directions, which caused splicing difficulties. An attempt in the spring of 1858 also failed when the cable broke due to an error. The ships returned to Devonport, and again took on more cable, sailing in late May. On 16 June 1858, HMS Agamemnon, 3200 tons, and USS Niagara, 5000 tons, two of the largest frigates in existence, met in mid-ocean, spliced the cable ends, and set out, Agamemnon for Valentia and Niagara for Trinity. On 17th August 1858 the first messages were exchanged, among them one between Queen Victoria and President Buchanan. The cable was worked at far too high a voltage (using an induction coil) on account of an erroneous theory, and gradually failed, becoming unusable after about 700 messages had been sent. Thomson's mirror galvanometer, invented in 1858, was used for receiving when the cable began to fail, though it was expected to work a relay initially. However, the practicality of an Atlantic cable had been firmly established.

An extensive and thorough inquiry was launched into the reasons for the failure and to establish principles for a new attempt. One thing that came out of this was the realization of the low conductivity of the copper then available. The success of the cable depended on as low a series resistance R, and as low a shunt conductance G, as possible. The usual commercial copper of the time gave as low as 40% of the conductivity of pure copper. Lake Superior copper was the best available, at 92% conductivity. The theory of the cable (which we shall treat here) had been studied by Professor William Thomson (1824-1907, later Lord Kelvin) of Glasgow University since 1855, when the subject of an Atlantic cable was first seriously considered. The very rapid rate of development should be noted. If anything, it was much more rapid than similar developments today.

A new attempt to lay the cable was delayed until 1865, because of financing difficulties during the American Civil War. The new cable had a core of 7 #18 Stubs' Gauge copper wire of 85% conductivity and weighing 300 lb/mile. The insulation was of 4 coats of gutta-percha, each coat sealed with Chatterton's Compound, weighing 400 lb/mile. The armor of steel wires made a cable 1.1" in diameter. This cable was laid by the gigantic Great Eastern, 22,000 tons, beginning on 30 June 1866, and requiring 14 days. A first cable broke not far from Trinity Bay. Another was laid successfully, while the first was fished up, spliced, and completed, so that two cables were then in service. Using a mirror galvanometer and double-current working, a signalling speed of 25 wpm was attained in that year. Thomson's siphon recorder was introduced in 1867, and capacitors for "curb" currents allowed more rapid signalling. It is remarkable that all this was done without the aid of electronic amplification. The success of the Atlantic cable caused the discontinuance of the construction of a land line connecting North America and Europe via Alaska and Siberia, which was already under way.

Another interesting long transmission line was the transcontinental telephone line from New York to San Francisco, placed in service in the summer of 1914. This was a pole line 3400 miles in length, which used six Shreeve mechanical repeaters and Pupin's loading coils. Repeaters were at Winnemucca, Salt Lake City, Denver, Omaha, Chicago and Pittsburgh. The mechanical repeaters were soon replaced by vacuum-tube repeaters using triodes, which gave improved signal quality with less maintenance. The 1861 transcontinental telegraph line, while an ambitious project and very important for communications, was not a technical challenge.

Although we will mainly use telephone lines as examples of transmission lines, the same theory applies to power lines, though the low frequency means that DC analysis can be used in most cases, and the speed of propagation neglected. The wavelength of 60 Hz power signals is 5000 km or 3107 miles, so even a quarter-wavelength line is much longer than most lines. Power transmission lines are almost all 3-phase open wire lines. Comments elicited by the great blackout of August 2003 brought up the speed of electrons, the public's knowledge extending to the existence of electrons as carrying electrical energy but not much further. If all the electrons in a copper conductor moved equally, a current of 100 A/cm^2 would only mean a drift velocity of 0.73 mm/s, a scarcely detectable creep. However, the electrons are rapidly moving about randomly, with velocities up to 1.56 x 10^6 m/s (3.5 million mph). In a current of 100 A/^{2}, a few of these fast electrons decide to move opposite to the direction of the electric field (electrons are negative, of course) rather than with it, and these few electrons are responsible for the current. Most of the electrons are totally indifferent. Of course, what is really important is not the motion of the electrons, but of energy, whether at a steady rate approaching the speed of light, or the group velocity of disturbances, which travel at the same rapid rate. Group velocity is discussed below.

Line Parameters

Before analyzing the transmission line, let's estimate the impedance parameters R, L, C and G, which are all specified per metre of line. This is not a trivial matter, and must be done with care if the calculations are to mean anything. The resistance R and inductance L comprise the series impedance Z, while the capacitance C and the leakage conductance G form the shunt admittance Y. Resistance and inductance depend on the frequency because of the skin effect, as does the part of G due to losses in the dielectric, while C is largely independent of frequency. G varies greatly with weather conditions on pole lines. The formulas for L and C, which may be found in Ramo, Whinnery and van Duzer, give the inductance and capacitance for fields outside the conductors. The impedance due to fields within the conductors must be added to L.

The skin depth $\delta = \sqrt{(\rho/\pi \mu f)}$ gives the penetration of fields in a plane conductor, amplitudes proportional to $e^{-z/\delta}$. This skin depth can also be applied to surfaces that are not too curved, such as wires, at least approximately. The surface resistance $R_s = \rho/\delta$ in ohms gives the resistance of a rectangular surface region as $R = R_s L/W$, where L is distance in the direction of the current, and W is the width. This is equivalent to assuming uniform current within the skin depth. For copper of resistivity 1.7×10^{-8} Ω-m, $\delta = 0.0660/\sqrt{f}$ and $R_s = 2.6 \times 10^{-7}\sqrt{f}$ Ω. For iron with an initial permeability of 250 and resistivity 9×10^{-8} Ω-m, $\delta = 0.00955/\sqrt{f}$ and $R_s = 9.424 \times 10^{-6}$ Ω. The AC resistance of a wire of diameter d is $R_s/\pi d$, while the DC resistance is $\rho(4L/\pi d^2)$. The boundary can be taken at the frequency for which $\delta = d/2$. The skin effect is important even at power frequencies, and cannot be neglected.

The internal inductance of a wire is $\mu/8\pi$, where μ is the permeability times $4\pi \times 10^{-7}$ H/m. It is remarkable that this does not depend on the size of the wire. For nonpermeable wire (copper or aluminium) $L_i = 5 \times 10^{-8}$ H/m, which is normally insignificant. For iron wire with an initial permeability of 250, $L_i = 1.25 \times 10^{-4}$ H/m, which will be dominant. However, the skin depth in iron wire is 6.6 times smaller than in copper at the same frequency, so in most cases the internal inductance can be neglected.

The typical telegraph wire was #6 BWG galvanized iron, or 0.203" = 5.156 mm diameter, with an area of 0.2088 cm^2. Therefore, its DC resistance is $R = 4.31 \times 10^{-3}$ Ω/m, or 4.31 Ω/km. Copper wire of the same diameter has R = 0.815 Ω/km, which is a factor of 5.3 less. Since early copper wire had a conductivity of only about 40%, the practical factor was closer to 2.1. This early wire was not made for electrical use, but for other purposes, and was pressed into service for telegraphy. It proved too weak for good wires, requiring low tension and large sag. BWG is Birmingham Wire Gauge, or Stubs' Gauge, commonly used before AWG, the American Wire Gauge, or Brown and Sharpe Gauge, was introduced in the 20th century.

Iron wire is an interesting example because of the internal inductance. The frequency for $\delta = 2.6 \times 10^{-3}$ m is f = 13.5 Hz. Below this frequency, we must use the internal impedance and find $L = 1.25 \times 10^{-4}$ H/m. The resistance R will be the DC resistance. Since we have a single wire, we need L and R only for it, assuming the earth perfectly conducting. If we assume that the wire is suspended at a height of 15 ft above the conducting earth, it is equivalent to a parallel-wire line with d = 0.5156 cm and s = 920 cm. For these measurements, $\cosh^{-1}(s/d) = 8.18$. Then, $C = \pi\varepsilon/8.18 = 3.40 \times 10^{-12}$ F/m and the external $L = 8.18(\mu/\pi) = 3.27 \times 10^{-6}$ H/m for two wires. For one wire, we find $C = 6.8 \times 10^{-12}$ F/m, L (external) = 1.64 x $^{-6}$ H/m. The wire is supported in air, so the dielectric constant is unity.

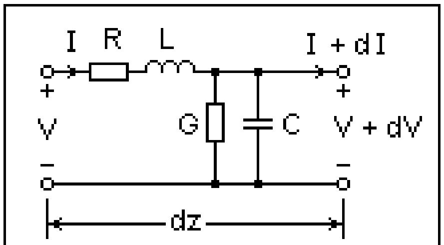
At high frequencies, the characteristic impedance will be $Z_o = \sqrt{(L/C)} = 491\Omega$, or about 250Ω wire to ground. The phase velocity will be $v = 1/\sqrt{(LC)} = 3 \times 10^8$ m/s, a velocity factor of unity. At low frequencies, $Z_o = 4300\Omega$, while $v = 6.85 \times 10^7$ m/s, a velocity factor of only 0.23. This variation of line parameters through an important frequency interval surely had an effect on communication over iron wires.

Forty poles per mile, the usual pole spacing, gives a span of 132 ft or 40.2 metres. If we assume that each insulator provides 10MΩ insulation, then $G = 10^{-7}/40.2 = 2.5 \times 10^{-9}$ S/m. This is a reasonable value for damp weather and a good line. In dry weather, values of 3000MΩ per mile are not uncommon. This gives $G = 2 \times 10^{-13}$ S/m. A reasonable average value might be $G = 10^{-10}$ S/m.

For a typical submarine cable, like the successful one of 1865, let's assume that the conductor is 7 #16 AWG (equivalent to #18 BWG), each of which has a resistance of 4Ω per 1000 ft. Then, $R = (4)/(0.85)(304.8)(7) = 2.2 \times 10^{-3}$ Ω/m, allowing for the 85% conductivity as specified. The frequency for a skin depth of 1.87 mm (the radius of the conductor bundle) is 125 Hz for copper. Since we are mainly interested in lower frequencies than this, we shall assume the DC resistance. The factor ln(r/r') = 1.812, where the outer radius is taken as 0.45 in (just inside the steel armor) and the inner radius is taken as (3/2)(.049) = 0.0735 in. This gives $C = \kappa(2\pi/1.812)\varepsilon = 7.98 \times 10^{-11}$ F/m, since $\kappa = 2.6$ for gutta-percha, and $L = (\mu/2\pi)(1.812) = 3.62 \times 10^{-7}$ H/m. Adding the internal inductance, we get $L = 4.12 \times 10^{-7}$ H/m. Finally, we make a wild guess at $G = 10^{-10}$ S/m. Better estimates of the parameters of the Atlantic cable have probably been made, but these are probably good enough for an example.

Let's consider a metallic telephone circuit composed of two #8 BWG copper conductors spaced at 12", as was used on the first transcontinental long-distance line. For this line, d = 4.19 mm, s = 304.8 mm, so s/d = 727.5 and $\cosh^{-1}(s/d) = 7.283$. At f = 988 Hz, the skin depth is 2.1 mm. The DC resistance would be $(2)(1.7 \times 10^{-8})/(1.38 \times 10^{-5}) = 2.46 \times 10^{-3}$ Ω/m. The AC resistance at about 10 kHz would be about 3.89×10^{-3} Ω/m. From the usual formulas, $C = \pi\varepsilon/7.283 = 3.82 \times 10^{-12}$ F/m and $L = (\mu/\pi)(7.283) = 2.91 \times 10^{-6}$ H/m. Note that the capacitance is actually less than that of the telegraph pole line, because in that case the earth formed a large plane electrode. For G we can again assume $G = 10^{-10}$ S/m.

Frequency Domain



Line Section

An equivalent circuit for a length dz of a uniform line, in which R, L, C and G are really Rdz, and so on, in terms of values per unit length, or for a single section of a line made up of discrete components where R, L, C and G are the component values and dz = 1, so that z is the node index. If the voltage and current vary slowly enough, a line with distributed values can be simulated by a line of discrete components, and vice-versa, with good accuracy. Discrete lines have some properties not shared by distributed lines, but we will not discuss them here, and they will not modify our results.

The differential equations describing the dependence of the voltage and current on time and space are linear, so that a linear combination of solutions is again a solution. This means that we can consider solutions with a time dependence $e^{j\omega t}$, and the time dependence will factor out, leaving an ordinary differential equation for the coefficients, which will be phasors depending on space only. Moreover, the parameters can be generalized to be frequency-dependent. Let $V(z,t) = V(z)e^{j\omega t}$ and $I(z,t) = I(z)e^{j\omega t}$. The positive directions of V and I are shown in the diagram.

We find that $dV = -(R + j\omega L)Idz = -ZIdz$ and $dI = -(G + j\omega C)Vdz = -YVdz$, or $dV/dz = -ZI$ and $dI/dz = -YV$. These first-order equations are easily uncoupled by a second differentiation, with the results: $d^2V/dz^2 - ZYV = 0$ and $d^2I/dz^2 - ZYI = 0$. Both V and I satisfy the same equation. Since ZY is independent of z and t, it can be represented by a constant $-k^2$. The minus sign is included so that k will appear as ±jkz in the exponential solutions of the equation. In fact, $V = V'e^{-jkz} + V''e^{jkz}$ will be the wave solution for V, and $I = (jk/Z)(V'e^{-jkz} - V''e^{jkz})$, from the first-order equation. $Z/jk = Z/\sqrt{(ZY)} = \sqrt{(Z/Y)}$ is the *characteristic impedance* Z_o. This is all quite simple if ZY is a real number, and we are already quite familiar with this case.

In general, ZY is not a real number, so k will have real and imaginary parts. If we write $k = k' - jk''$, the wave $V'e^{j(\omega t - kz)}$ will be $V'e^{-k''z}e^{j(\omega t - k'z)}$, an exponentially-damped sinusoidal wave. The real part k' is the *phase constant*, while the negative imaginary part k'' is the *attenuation coefficient*. The wave in the opposite direction is also exponentially damped in the direction of propagation with the same k''.

Now $-k^2 = -(k' - jk'')^2 = (R + j\omega L)(G + j\omega C)$ or $k'^2 - k''^2 = \omega^2LC - RG$, and $2k'k'' = \omega(LG + CR)$. We substitute for k'' in the first equation by its value in the second, and solve the resulting quadratic in k'^2. This gives us $k'^2 = (1/2)\{(\omega^2LC - RG) + \sqrt{[(\omega^2LC - RG)^2 + \omega^2(LG + CR)^2]}\}$. The (-) sign before the radical is an unacceptable solution. Therefore, we always get two values ±k' corresponding to waves travelling in opposite directions. Having found k', we then find $k'' = (\omega/k')(LG + CR)/2$. Note that ω/k' is the phase velocity v, and that CR has the dimensions s/m^2. If we define a time constant τ by $LG + CR = 1/v^2\tau$, then $e^{-k''z}$ becomes $e^{-t/\tau}$.

For any frequency ω, these equations will give us values for k' and k'' with reasonable ease. However, plain numbers are not very illuminating, so we examine some special limiting cases that are often found in practice. First, suppose that RG can be neglected in comparison with ω^2LC. At a high enough frequency, this will always be the case. In this case, we find $k'^2 = \omega^2LC + (\tau/4\omega)^2$ to lowest order in ω, using the quantity τ defined in the preceding paragraph. If the second term can be neglected, then we have $k'^2 = \omega^2LC$, the dispersion relation for a lossless line. However, we now do have an attenuation, and $k'' = [1/\sqrt{(LC)}](LG + CR)/2$. Waves with frequency components lying only in this range (which can be extensive) will propagate without distortion, since the velocity and attenuation are the same for all frequencies. Most practical transmission lines are intended to operate in this region, if possible.

At very low frequencies on a lossy line, we can neglect ω^2LC in comparison with RG. This gives k' = ω (LG + CR)/√(RG) and k" = √(RG)/2. The phase velocity is ω/k' = √(RG)/(LG + CR), and is independent of frequency. The attenuation is also frequency-independent, so waves propagate without distortion.

Consider a very special line in which L/R = C/G = x. Then $k'^2 = (RG/2)\{\omega^2x^2 - 1 + \sqrt{[(\omega^2x^2 + 1)^2]}\}$ or, quite simply, $k'^2 = \omega^2LC$, the same as for a line with R = G = 0! The attenuation constant is k" = RC/√L. We have nondispersive propagation with, in general, a low attenuation. This distortionless line was recognized by Oliver Heaviside in 1893, and was the subject of the famous patent of M. I. Pupin of 19 June 1900 that made long-distance telephony possible. More will be said about this later.

Heaviside, for some reason, called a line for which L = G = 0 an "ideal line," a term usually reserved for a lossless line. We have to go back to the beginning for the analysis, and find that V satisfies d^2V/dt^2 - jωRC V = 0. From $-k^2$ = jωRC, we find that k' = k" = √(ωRC/2). The phase velocity is v = √(2ω/RC), and the group velocity v_g = dω/dk' = 2v. These strongly attenuated waves are somewhat like surface waves in deep water, where the group velocity is half the phase velocity. The frequency domain analysis tends to obscure what is really taking place here, which will be clarified in the time domain.

To show how to use the equations derived in this section, let us consider an open-wire metallic circuit of two #12 AWG conductors spaced 12" apart. This was a very typical open-wire telephone line, and we have good figures for the parameters, which were given as G = 0.29 μ S/mile, R = 17.1 Ω/mile, L = 3.73 mH/mile and C = 7.83 nF/mile, all very characteristic figures typical of dry weather. In consistent units per metre, these are G = 1.80 x 10^{-10} S/m, R = 1.06 x 10^{-2} Ω/m, L = 2.32 x 10^{-6} H/m and C = 4.87 x 1-^{-12} F/m. The first step is to form the products LC, RG and LG + CR. Then, for the specified angular frequency ω, ω^2LC - RG is found. This is squared, added to the square of ω(LG + CR), the square root is taken. The sum of this and the first expression, divided by 2, is k'^2. The square root is taken, giving k', and then the phase velocity ω/k' = v is found. The attenuation k" = v(LG + CR)/2. The attenuation coefficient in dB/mile is 20 log exp(-1609k") = 1.40 x 10^4k". To find the characteristic impedance Z_o, Z = R + jωL and Y = G + jωC are first calculated. Using the HP-48 or equivalent, the polar forms are determined, divided, and the square root is taken. With the help of a calculator, these operations are straightforward and can be done without much trouble. Of course, they can be programmed and the results obtained at the press of a button.

For our example, the parameters were found at 300 Hz, 1000 Hz and 3000 Hz. At 300 Hz, k' = 0.847 x 10^{-5}, giving v = 2.23 x 10^8 m/s or 0.74c. k" = 5.79 x 10^{-6}, or -0.081 dB/mile. The characteristic impedance is 1345/_-36.81° Ω. At mid-band, 1000 Hz, k' = 2.231 x 10^{-5}, and v = 2.82 x 10^8 m/s = 0.94c. Z_o = 841/_-23.68° and k" = 7.33 x 10^{-6} = -0.102 dB/mile. At 3000 Hz, k' = 6.383 x 10^{-5}, and v = 2.95 x 10^8 m/s or 0.98c. Z_o = 712/_-10.01° and k" = 7.68 x 10^{-6} = 0.107 dB/mile.

From these figures, we note that the attenuation is roughly independent of freqency and quite low. A 200-mile line will have an attenuation of only about 20 dB. The characteristic impedance is high and capacitive, around 1 kΩ in magnitude. Phase velocity dispersion is large, with $\Delta v/v = 0.25$, approximately. Since the ear is not sensitive to phase distortion, the dispersion has little effect because of the constant attenuation. Over long distances, this can be corrected by the phase response of the repeaters. The dispersion can be greatly reduced by loading the lines (as will be explained below), but since this was not required after vacuum-tube repeaters were introduced, open-wire lines were not loaded in the United States in later years.

Time Domain

The problem is altogether more difficult in the time domain, though the results are much more illuminating. The parameters must now be true constants, independent of frequency. This is, at best, an approximation. Writing the circuit equations in the time domain, we find $\partial V/\partial z = -RI - L\partial I/\partial t$, and $\partial I/\partial z = -GV - C\partial V/\partial t$. Again, we can separate the equations by taking the second derivatives, and show that V and I satisfy the same equation. For V, it is $\partial^2 V/\partial z^2 = LC\partial^2 V/\partial t^2 + (LG + CR)\partial V/\partial t + RGV$.

This equation is called the *telegraph* or *telegrapher's* equation, first studied by William Thomson in connection with the Atlantic cable in 1855. It is a second-order elliptic partial differential equation whose solution is rather difficult. If the voltage and current are known at t = 0 for all z, the solution can be obtained from formulas derived by Riemann and Volterra, which was first done by duBois Reymond in 1889. The form of the solution is a solution of the wave equation, which the telegrapher's equation becomes in the lossless case, plus a *residual wave* that is left behind after a pulse passes, and decreases exponentially with time. The details, which are rather complicated, can be found in the References.

The first-order time derivative can be removed by means of a simple substitution. Let $V = u\ e^{-[(LG + CR)/2LC]t}$, and substitute this in the equation. The result, after considerable annoying but straightforward algebra, is $\partial^2 u/\partial t^2 = (1/LC)\partial^2 u/\partial t^2 + (R/L - G/C)u$. Note that we put the t derivative on the left and the z derivative on the right. This is simply the wave equation plus a rude contribution from u on the right. It is easy to see that if R/L = G/C, this term vanishes and we have the simple wave equation. This confirms the properties of lines of this special type that we mentioned in the frequency domain (and, in fact, suggested it). The u term is the source of the residual waves, which are opposite in phase coming from R and G, which accounts for the cancellation.

For Heaviside's "ideal line" with L = G = 0, the telegraph equation becomes $\partial^2 V/\partial z^2 = CR(\partial V/\partial t)$. This doesn't look much like the wave equation, since the second derivative with respect to time is absent, but it is recognizable as a diffusion equation, such as the heat conduction equation, $\partial^2 T/\partial z^2 = k(\partial T/\partial t)$, where the diffusivity k = K/cd, where K is the heat conductivity, c the specific heat per unit mass, and d is the density. The solution of this equation for T = 0 initially, and with the face z = 0 raised to temperature T at t = 0, is $T(z,t) = T\ \mathrm{erfc}[z/2\sqrt{(kt)}]$. erfc(x) is the complementary error function, 1 - erf(x), where erf(x) = (2/√&pi:)∫(0,x)exp(-u ^{2})du is a tabulated function, and erf(∞) = 1. We can adapt this solution by simply setting T = V and k = RC. Voltage "diffuses" into a line of this type.

If we apply a voltage V suddenly to the end of a long line of this type at t = 0, let us say that the signal begins arriving when its amplitude is 0.1V, is fully arrived when its amplitude is 0.9V, and "arrives" when its amplitude is 0.5V. The corresponding values of x in the error function are 1.17, 0.09 and 0.48. The pulse arrives at a distance z at time t = 1.08 RCz^2, and its rise time is 30.7 RCz^2. When these times are less than those corresponding to propagation at the velocity of light, they cannot be true. A small inductance L must be present. At distances where they are much later than light-speed arrival times, they are more valid. This analysis is the basis for Kelvin's "KR Law" (capacitance was then denoted by K) that the arrival time was proportional to the square of the distance and proportional to CR.

For the Atlantic cable, we estimated RC = 1.76 x 10^{-13} s/m^2. The length of the cable is 3.039 x 10^6 m, so RCz^2 = 1.62 s. The arrival time would then be 1.75 s, and the rise time 50 s! One would not wait for the full rise time, however, but apply a reverse current to discharge the line.

Telephone Cables and Loading

Early telephones used the same grounded iron-wire pole lines as the telegraph. They were cheap and strong, and gave good transmission properties over the distances then in use. Unfortunately, they also were plagued with crosstalk (since telephone lines were concentrated far more than telegraph lines, especially near central offices), and from earth-current interference due to street lighting and electric railways in cities, where most early telephones were installed. Lines usually ran over rooftops, which made an inferior line, or in large numbers on poles. Tall poles in West Stret, New York, had 25 crossarms, each carrying 10 wires, for 250 wires in all.

Grounded iron-wire pole lines were much less objectionable in rural areas, so they survived there for a very long time. Grounded iron-wire lines were also still common for short lines of 2 miles or so in cities in 1900. However, these lines were unsuitable for long-distance service over distances of 100 miles or more. Hard-drawn copper wire appeared about 1883, and rapidly supplanted iron wire for all important circuits, with its combination of conductivity and strength. Metallic circuits on pole lines, with two copper wires spaced 12" apart, transposed once or twice a mile to reduce crosstalk, were far superior to earlier lines, and were used, for example, in the first transcontinental line of 1914, and for the first carrier circuits in 1918.

Crosstalk is the appearance of signals in neighboring circuits, which is very much more annoying with the telephone than "crossfire" was with the telegraph, which is very resistant to interference. It can be caused by such things as common impedances or batteries, but in telephone circuits it is due to stray fields. Magnetic coupling immediately comes to mind, but this is very small in telephone circuits. Much more important was electrostatic coupling, the induction of charges in a neighboring circuit due to charges in the interfering circuit. The predominance of electrostatic coupling was pointed out by J. J. Carty in 1889. Carty was an early proponent of metallic circuits for telephones (1881), the inventor of the common-battery switchboard (1893) and of the phantom circuit (see below). Transposition of line wires at regular intervals can be used to reduce electrostatic coupling, and this was always done with telephone open-wire lines. Transposition was introduced by Barrett in 1885.

Such open-wire circuits were very inconvenient in cities, where many circuits came together. At an early date, *twisted pair* was introduced, of cloth, paper or fibre insulated wire twisted about three times in an inch. The inductance was decreased, the capacitance was increased, while resistance and conductance remained about the same. Crosstalk with neighboring circuits was greatly reduced because of the twisting and close spacing of the conductors. Rubber insulation was found unsuitable for telephone use, though it was widely used in electric power. A large number of twisted pairs was assembled into a *cable* and armored with a sheath of 97 Pb - 3 Sn alloy about 1/8" thick, applied by pressure. The lead positively excluded water, which was extremely ruinous to cables, while remaining flexible enough for easy installation. Paraffin wax impregnated the cable ends to assure sealing against water. The cables ended in *cable heads*, from which the individual circuits could be led away to a terminal board. These cables were not only used for trunk circuits in central offices, but were buried in conduit with access manholes beneath the streets and brought up into cable heads at intervals. Telephone service is still distributed in this way, though materials and manufacturing processes have changed. Cables are now often suspended on pole lines, often the same ones used for electricity supply, and are frequently seen along streets.

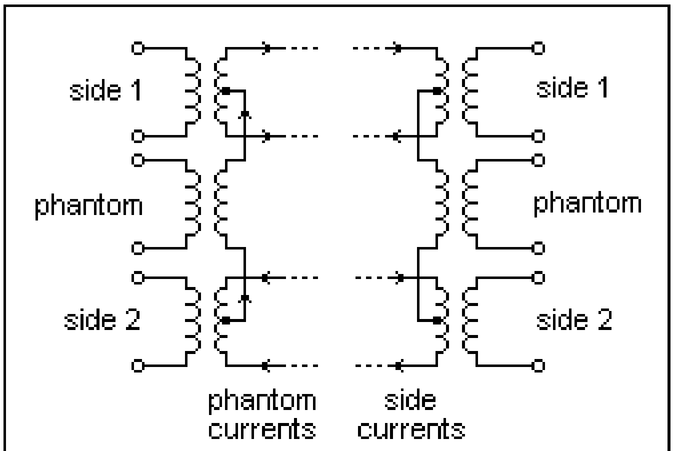
In 1892, cables with 100 pairs of #19 AWG wire were in service. By 1901, 404 pairs of #22 AWG were available in one cable, by 1914 1212 pairs of #24 AWG, and in 1939, 2121 pairs of #26 AWG. By 1937, the Bell System had 6.7 million miles of cable, 2 million miles of open wire, and about 750,000 miles of carrier in long-distance service. Microwave carrier systems came in after World War II, and optical fibre cables towards the end of the century. Except for subscriber loops, most telephone circuits are now 4-wire circuits, with independent half-duplex circuits for the two directions, instead of a 2-wire full duplex circuit. This greatly simplifies interfacing and repeaters, but would have been thought an extravagance in earlier days.

We now return to the story of how open-wire lines and cables were made practical for long-distance transmission. The story begins before the age of electronic amplification, so it was essential to reduce the attenuation to an absolute minimum, while keeping frequency distortion to a minimum. Frequency distortion, the result of dispersive propagation, was perhaps even more important than attenuation. Over a long line, a signal could often be heard, but it was so mangled it was incomprehensible. In 1893 Oliver Heaviside proposed lines in which CR = LG. Such lines were distortionless, and the attenuation was a minimum. In most cases, CR is much less than LG, or G/C much less than R/L. G could be increased, but this would also increase attenuation. C could not be made any smaller easily; minimum C was already a desideratum. R, likewise, could not easily be reduced. The only alternative was to increase L.

The early iron-wire lines were probably improved by the permeability of the iron, not impaired, as was commonly thought. The skin effect, as we have pointed out above, probably reduced the effect of the permeability to a negligible level. Adding L to a line is called *loading*, and can be either *continuous* or *discrete*. Continuous loading is done by wrapping the wire with wire or tape of a permeable metal. Iron was first used, but *Permalloy* ("Mumetal"), a high-Co Co-Fe alloy, sometimes with added Mo; or *Perminvar*, 45 Ni - 30 Fe - 25 Co; or *Permendur*, 50 Fe - 50 Co were later used as the alloys were developed. The conductor is wrapped with a thin tape to reduce eddy current losses.

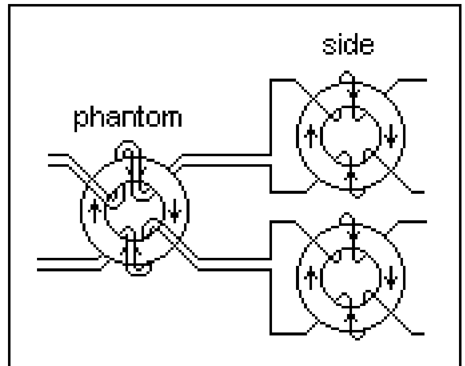
Discrete loading was done with *loading coils* with toroidal cores. In 1916, powdered iron cores replaced iron wire bundles. In 1926, powdered Permalloy, and later powdered Mo-Permalloy, gave smaller and smaller cores. These coils added about 100 mH of inductance per section. The standard spacings of loading coils were denoted by letters. M was 9000 ft or 2740 m, H was 6000 ft or 1830 m, and B was 3000 ft or 914 m. Loading was denoted by, for example, H-88 for 88 mH every 6000 ft. The value of the inductance applies to both wires, and is the total amount added to the circuit per length S, the spacing of the coils. Both conductors were wound around the cores so that the equal and opposite currents aided, and the circuit remained balanced.

The proper amounts and spacing of loading were not as easy to determine as it might seem, and there were early failures in applying Heaviside's recommendations. The correct methods were worked out by M. I. Pupin and G. A. Campbell, working independently, about 1900. Pupin was granted priority, but Campbell's contributions should not be neglected. Before the age of vacuum-tube amplifiers, loading was essential to reduce the attenuation of long-distance circuits.



Phantom Circuit

The typical open-wire line element consisted of four wires, which provided two *side circuits* and one *phantom circuit*, as well as telegraph circuits in many cases. The idea of the phantom circuit is shown in the diagram. All that is necessary is audio transformers with center-tapped secondaries. It is easy to set up in the laboratory and observe. Note that these circuits do not carry DC, which is necessary only on subscriber lines. Phantom circuits were easy to arrange and were economical, so they were very widely used. Since the phantom currents are the same in each wire of a side circuit, and opposite in direction in the two side circuits, it was possible to load the phantom as well as the side circuits. This method of loading is called the Campbell-Shaw method, as shown at the right.



Campbell-Shaw Loading

As an example, we shall consider a paper-insulated twisted pair cable with #19 AWG conductors. #19 wire has a DC resistance of 43Ω/mile. The skin depth for copper at 1 kHz is 2.1 mm, while #19 wire has a diameter of 0.91 mm. Therefore, we can neglect the skin effect and take R = 86Ω/mile or 5.34 x 10^{-2} Ω/m. From published data on this cable, G = 1.4 μ S/mile, or G = 8.70 x 10^{-10} S/m, L = 1 mH/mile or 6.2 x 10^{-7} H/m, and C = 0.062 μ F/mile, or 3.85 x 10^{-11} F/m. For this cable, G/C = 22.6 and R/L = 86,100, so these quantities are by no means equal. In the frequency domain, our equations give k' = 8.32 x 10^{-5} per m, and k" = 7.78 x 10^{-5} per m, at f = 1 kHz. The phase velocity v = 7.55 x 10^{7} m/s, or 0.25c. The attenuation is 20 log($e^{-k''}$) = -1.09 dB per mile (1 mile = 1609 m).

The side circuits are given H-172 loading, which adds 172 mH every 6000 ft to the circuit, and 13.6Ω. R is now 6.09 x 10^{-2} Ω/m and L becomes 9.45 x 10^{-5} H/m. C and G remain unchanged. Now R/L is 644, a much closer match to G/C. k' becomes 3.813 x 10^{-4} per m, so the phase velocity is only 1.65 x 10^{7} m/s, or

0.055c. On the other hand, k" = 2.01×10^{-5}, so that the attenuation is now only -0.28 dB/mile, a factor of 4 less than that of the unloaded cable. More importantly, distortion is now very much less. The attenuation would only be 28 dB in a 100-mile cable. The first long loaded submarine telephone cables were the three 100-mile cables laid between Key West and Havana in 1921 in up to 1000 fathoms. They used iron-wire loading. The phantom circuit loading, incidentally, was H-63 for the cable we have been considering. This example illustrates the great benefits of loading. All submarine cables, and many land cables, are still loaded. Loaded open-wire lines disappeared before World War II, as cables took over.

Discrete loading brings another effect with it, that of a *cutoff frequency* for the line. Each section with its loading coils looks like a section of a low-pass filter. The cutoff frequency of this filter is $f = 1/\pi\sqrt{(LSC)}$. SC is the total capacitance between loading coils, and L is the loading inductance. For the loaded cable we have been considering, SC = 7.04×10^{-8} F, and L = 0.172 H. This gives f = 2900 Hz. This is at the upper end of the telephone passband. Continuous loading does not exhibit this effect.

At 1 kHz, the series impedance of the unloaded cable is Z = $5.34 \times 10^{-2} + j3.90 \times 10^{-3}$ Ω, and the shunt admittance is Y = 8.70×10^{-10} S. The characteristic impedance is $\sqrt{(Z/Y)}$ = 470/_-43° Ω. The loaded cable has Z = $5.34 \times 10^{-2} + j0.594$, which changes the characteristic impedance to 1569/_-2.47° Ω, which is not only four times larger, but is also nearly real. For the loaded line, this is not far from $Z_o = \sqrt{(L/SC)}$, where L, S and C have the same meanings as in the cutoff frequency formula.

Coaxial Cables

The fields of open-wire lines cause crosstalk between neighboring circuits, and the pickup of interference from external sources. At high frequencies, such circuits will even lose energy by radiation. All these problems are exacerbated at high frequencies, so open-wire lines are suitable only for lower frequencies, below, say 100 kHz. Stray fields are very much reduced with twisted pairs, and the lead armor of cables is perfect shielding from external sources. Shielded twisted pairs are often used to reduce interference. It should be remembered that the shielding is useless unless it is grounded.

Perfect shielding also results if one conductor completely surrounds the other. If the outer conductor is grounded, then induced charges on its outer surface shield the interior without any internal effect. Such lines are called *coaxial*, and may be rigid or flexible. The flexible lines are *coaxial cables*. The Atlantic cable was a coaxial transmission line, so coaxial cables are not a new thing. More recently, they have been used at high (radio) frequencies, where radiation losses had to be avoided. The lines are generally short ones, so the high attenuation is inconsequential. We have seen that at high frequencies, R and G become less important relative to L and C. Then, $v = 1/\sqrt{(LC)}$ becomes practically constant so phase distortion is small, and $Z_o = \sqrt{(L/C)}$, which is approximately real. These are the excellent characteristics of an ideal transmission line. A disadvantage of coaxial lines is that they are *unbalanced*, meaning that the two conductors are not electrically symmetrical, as are the two conductors of a parallel-wire line.

The centre conductor has to be supported. This can be done by insulating spacers at intervals, which keeps the capacitance low, but also introduces an undesirable discrete inhomogeneity, which can result in filter action. To avoid this, the dielectric is usually made continuous. A typical coaxial cable has a center conductor that may be solid, surrounded by a dielectric such as polyethylene (PE) that has replaced gutta-percha, with a layer of copper braid over this, which serves as the outer conductor. A protective sheath then covers all. The capacitance per metre of cylindrical coaxial cable is C = 2πε/ln(b/a) and the inductance per metre is L = (μ /2π)ln(b/a), where b is the inside diameter of the outer conductor, and a in the outside diameter of the inner conductor. ε is κ(8.854×10^{-12}) F/m and μ = 4π x 10^{-7} H/m. For PE, κ = 2.3. Coaxial lines have low characteristic impedances, typically from 30 to 100 Ω.

Attenuation in such a coaxial cable is mainly due to losses in the dielectric and increases with the frequency. The attenuation due to resistance in the conductors is k'' = $R/2Z_o$. Of course, skin effect must be considered. The attenuation due to the dielectric is (k'/2)(κ''/κ'), in terms of the imaginary and real parts of the dielectric constant. For polystyrene, the ratio κ''/κ is 0.7 x 10^{-4} at 1 MHz, 10^{-4} at 100 MHz and 4.3 x 10^{-4} at 10 GHz. For comparison, the values for rubber are shockingly high (about 0.1). For ordinary coaxial cables, the attenuation varies about as √f. The very widely used 0.405" OD RG-8/U has Z_o = 52Ω, k'' = 6.8 x 10^{-4} per metre at 1 MHz, and v = 0.66c.

Coaxial cable exhibits its desirable properties at high frequencies. At low frequencies, it is worse than open-wire line in most respects, due to its high capacitance and about three times higher attenuation. At any frequency, it retains its excellent shielding. Its use in oscilloscope probes is made possible by the 10x probe, which neutralizes the effect of its large capacitance by sacrificing gain. Coaxial cable is excellent for carrier lines, in which many voice channels are multiplexed onto one high-frequency carrier signal. The low phase distortion is necessary for this application. "Coax" is used in the "cable connections" that bring satellite channels into the home for just this reason.

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